

A QUILLEN THEOREM B_n FOR HOMOTOPY PULLBACKS

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ABSTRACT. We prove an extension of the Quillen Theorem B_n for homotopy *fibres* of [DKS, §6] to a similar result for homotopy *pullbacks* and use this to obtain sufficient conditions on a zigzag $\mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z}$ between categories in order that *its pullback is a homotopy pullback*.

1. INTRODUCTION

1.1. The background. In [Q, §1] Quillen proved his Theorem B which gave a rather simple description of the homotopy fibres of a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ if f had a certain *property* B_1 .

This was generalized in [DKS, §6] where it was shown that increasingly weaker *properties* B_n ($n > 1$) allowed for increasingly less simple descriptions of these homotopy fibres. Moreover it was noted that a sufficient condition for a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ to have property B_n ($n > 1$) was that the category \mathbf{Y} has a certain *property* C_n .

1.2. The current paper. We show that for a zigzag $f: \mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z} : g$ in which f has property B_n (1.1) (and in particular if \mathbf{Y} has property C_n (1.1)), its homotopy pullback admits a description rather similar to the ones mentioned in 1.1.

Moreover the pullback $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ of this zigzag comes with a monomorphism into this homotopy pullback and hence is itself a homotopy pullback if the monomorphism is a weak equivalence.

1.3. The motivation. Our result (1.2) and in particular its second half is what really motivated us to write the present note, and well for the following reasons.

In [R, 8.3] Charles Rezk proved that

- for every *simplicial model category* one (and hence every) Reedy fibrant replacement of its simplicial nerve is a *complete Segal space*.

Although the proof of the Segal part of this result relied heavily on the simplicial structure, it seemed that this result would also hold without the assumption of a simplicial structure.

In fact as we will show in [BK], most of the model structure is superfluous. All that is needed is that there is a category of weak equivalences with three rather simple properties. More precisely we will show that

- Charles Rezk's result holds for every *relative category* which has the *two out of six property* and admits a *3-arrow calculus*.

It turns out that in that situation the category of the weak equivalences has property C_3 with the result that, in view of our result (1.2) for $n = 3$, the verification of the Segal property, i.e. showing that certain fibre products (which are iterated pullbacks) are homotopy fibre products (which are iterated homotopy pullbacks), is reduced to a rather simple calculation.

1.4. The proof. The homotopy fibre results of (1.1) were obtained by an induction on n which at each stage used Quillen's Theorem B.

To prove our homotopy pullback results (1.2) it turns out to be convenient to go one step further back to the lemma that Quillen used to prove his Theorem B and which can be summarized as follows:

- If $F: \mathbf{D} \rightarrow \mathbf{Cat}$ is a \mathbf{D} -diagram of categories and *weak equivalences* between them, $\mathbf{Gr} F$ its Grothendieck construction and $\pi: \mathbf{Gr} F \rightarrow \mathbf{D}$ the associated projection functor, then, *for every object $D \in \mathbf{D}$, the fibre*

$$\pi^{-1}D = FD$$

of π over D is also a homotopy fibre.

Using this result we first give a different non-inductive proof of the results of (1.1) and then note that this proof almost effortlessly extends to a proof of the homotopy pullback results of (1.2).

1.5. Organization of the paper. There are three more sections.

In the first (§2) we discuss various *Grothendieck constructions* and give a precise formulation of what we will call *Quillen's lemma*.

In the next section (§3) we recall the properties B_n and C_n and state the *Theorems B_n for homotopy fibres* and for *homotopy pullbacks*.

The last section (§4) then is devoted to a proof of these two Theorems B_n .

2. PRELIMINARIES

In preparation for the formulation and the proofs of our results we here

- briefly discuss Grothendieck constructions,
- formulate, in terms of Grothendieck constructions, a categorical version of the lemma that Quillen used in his proof of Theorem B, and
- describe three Grothendieck constructions which will be used in our proofs.

But first a comment on

2.0. Terminology. We will work in the category \mathbf{Cat} of small categories with the Thomason model structure [T2] in which a map is a *weak equivalence* iff its *nerve* is a weak equivalence of simplicial sets and in which *homotopy fibres* and *homotopy pullbacks* have a similar meaning.

2.1. Grothendieck constructions. Given a small category \mathbf{D} and a functor $F: \mathbf{D} \rightarrow \mathbf{Cat}$ (2.0), the **Grothendieck construction** on F is the category $\mathbf{Gr} F$ which has

- (i) as *objects* the pairs (D, A) consisting of objects

$$D \in \mathbf{D} \quad \text{and} \quad A \in FD \ ,$$

- (ii) as *maps* $(D_1, A_1) \rightarrow (D_2, A_2)$ the pairs (d, a) of maps

$$d: D_1 \rightarrow D_2 \in \mathbf{D} \quad \text{and} \quad a: (Fd)A_1 \rightarrow A_2 \in FD_2$$

and

- (iii) in which the *composition* is given by the formula

$$(d', a')(d, a) = (d'd, a'((Fd)a)) \ .$$

Moreover

- (iv) $\mathbf{Gr} F$ comes with a *projection functor* $\pi: \mathbf{Gr} F \rightarrow \mathbf{D}$ which sends an object (D, A) (resp. a map (d, a)) in $\mathbf{Gr} F$ to the object D (resp. the map d) in \mathbf{D} .

The usefulness of Grothendieck constructions is due to the following property which was noticed by Bob Thomason [T1, 1.2]:

2.2. Proposition.

- (i) *The Grothendieck construction is a homotopy colimit construction on the category \mathbf{Cat} ,*

and hence

- (ii) *it is homotopy invariant in the sense that every natural weak equivalence (2.0) between two functors $F_1, F_2: \mathbf{D} \rightarrow \mathbf{Cat}$ induces a weak equivalence $\mathbf{Gr} F_1 \rightarrow \mathbf{Gr} F_2$.*

Next we note that Quillen's key observation in the lemma that he used to prove Theorem B was that certain functors $\mathbf{D} \rightarrow \mathbf{Cat}$ had what we will call

2.3. Property Q. Given a small category \mathbf{D} , a functor $F: \mathbf{D} \rightarrow \mathbf{Cat}$ will be said to have **property Q** if it sends all maps of \mathbf{D} to weak equivalences in \mathbf{Cat} .

A categorical version of the lemma that Quillen used in the proof of Theorem B (a proof of which can be found in [GJ, IV, 5.7]) then becomes in view of 2.2(i) above

2.4. Quillen's lemma. *If, given a small category \mathbf{D} , a functor $F: \mathbf{D} \rightarrow \mathbf{Cat}$ has property Q (2.3), then, for every object $D \in \mathbf{D}$, the fibre*

$$\pi^{-1}D = FD$$

of π (2.1(iv)) over \mathbf{D} is a homotopy fibre.

It remains to construct the promised three Grothendieck constructions.

We start with

2.5. Two Grothendieck constructions associated with a functor $\mathbf{X} \rightarrow \mathbf{Y}$.

Given an integer $n \geq 1$ and a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ between small categories, we denote by $(f\mathbf{X} \downarrow_n \mathbf{Y})$ the category of which

- (i) an object consists of a pair of objects

$$X \in \mathbf{X} \quad \text{and} \quad Y \in \mathbf{Y}$$

together with an alternating zigzag

$$fX = Y_n \cdot \cdot \cdot Y_2 \longleftarrow Y_1 \longrightarrow gZ \quad \text{in } \mathbf{Y}$$

and of which

(ii) a *map* consists of a pair of maps

$$x: X \rightarrow X' \in \mathbf{X} \quad \text{and} \quad y: Y \rightarrow Y' \in \mathbf{Y}$$

together with a commutative diagram

$$\begin{array}{ccccccc} fX = Y_n & \cdots & Y_2 & \longleftarrow & Y_1 & \longrightarrow & Y \\ fX \downarrow & & \downarrow & & \downarrow & & \downarrow y \\ fX' = Y'_n & \cdots & Y'_2 & \longleftarrow & Y'_1 & \longrightarrow & Y' \end{array} \quad \text{in } \mathbf{Y}$$

(iii) This category comes with a monomorphism

$$h: X \longrightarrow (f\mathbf{X} \downarrow_n \mathbf{Y})$$

which sends each object $X \in \mathbf{X}$ to the zigzag of identity maps which starts at fX .

Furthermore

(iv) let, for every object $Y \in \mathbf{Y}$

$$(f\mathbf{X} \downarrow_n Y) \subset (f\mathbf{X} \downarrow_n \mathbf{Y})$$

denote the subcategory consisting of the objects which end at Y and the maps which end at $!_Y$

and similarly

(v) let, for every object $X \in \mathbf{X}$

$$(fX \downarrow_n \mathbf{Y}) \subset (f\mathbf{X} \downarrow_n \mathbf{Y})$$

denote the subcategory consisting of the objects which start at fX and the maps which start at 1_{fX} .

The naturality of $(f\mathbf{X} \downarrow_n Y)$ and $(fX \downarrow_n \mathbf{Y})$ in respectively Y and X then readily implies

2.6. Proposition. *For every integer $n \geq 1$ and functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ between small categories (2.1)*

$$(i) \quad (f\mathbf{X} \downarrow_n \mathbf{Y}) = \mathbf{Gr}((f\mathbf{X} \downarrow_n -): \mathbf{Y} \rightarrow \mathbf{Cat})$$

and

$$(ii) \quad (f\mathbf{X} \downarrow_n \mathbf{Y}) = \begin{cases} \mathbf{Gr}((f- \downarrow_n \mathbf{Y}): \mathbf{X} \rightarrow \mathbf{Cat}) \\ \text{or} \\ \mathbf{Gr}((f- \downarrow_n \mathbf{Y}): \mathbf{X}^{\text{op}} \rightarrow \mathbf{Cat}) \end{cases}$$

depending on whether n is even or odd.

We end with

2.7. A Grothendieck construction associated with a zigzag $\mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z}$.

Given an integer $n \geq 1$ and a zigzag $f: \mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z} : g$ between small categories, we denote by $(f\mathbf{X} \downarrow_n g\mathbf{Z})$ the category of which

(i) an *object* consists of a pair of objects

$$X \in \mathbf{X} \quad \text{and} \quad Z \in \mathbf{Z} ,$$

together with an alternating zigzag

$$fX = Y_n \cdots Y_2 \longleftarrow Y_1 \longrightarrow gZ \quad \text{in } \mathbf{Y}$$

and of which

- (ii) a map consists of a pair of maps

$$x: X \rightarrow X' \in \mathbf{X} \quad \text{and} \quad z: Z \rightarrow Z' \in \mathbf{Z} ,$$

together with a commutative diagram

$$\begin{array}{ccccccc} fX = Y_n & \cdots & Y_2 & \longleftarrow & Y_1 & \longrightarrow & gZ \\ \downarrow fx & & \downarrow & & \downarrow & & \downarrow gz \\ fX' = Y'_n & \cdots & Y'_2 & \longleftarrow & Y'_1 & \longrightarrow & gZ' \end{array} \quad \text{in } \mathbf{Y} .$$

- (iii) This category comes with a monomorphism

$$K: (\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}) \longrightarrow (f\mathbf{X} \downarrow_n g\mathbf{Z})$$

which sends each object $(X, Z) \in \mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ to a zigzag of identity maps starting at fX and ending at gZ .

Furthermore

- (iv) we denote, for every object $Z \in \mathbf{Z}$, by

$$(f\mathbf{X} \downarrow_n gZ) \subset (f\mathbf{X} \downarrow_n g\mathbf{Z})$$

the subcategory consisting of the objects which end at gZ and the maps which end at 1_{gZ} .

The naturality of $(f\mathbf{X} \downarrow_n g\mathbf{Z})$ with respect to Z then readily implies

2.8. Proposition. For every integer $n \geq 1$ and zigzag $f: \mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z}: g$ between small categories (2.1)

$$(f\mathbf{X} \downarrow_n g\mathbf{Z}) = \mathbf{Gr}((f\mathbf{X} \downarrow_n g-): \mathbf{Z} \rightarrow \mathbf{Cat}) .$$

3. THE RESULTS

We start with recalling the homotopy fibre results of (1.1), beginning with the notion of

3.1. Property B_n . Given an integer $n \geq 1$, a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ between small categories is said to have **property B_n** if the functor (2.6(i))

$$(f\mathbf{X} \downarrow_n -): \mathbf{Y} \longrightarrow \mathbf{Cat}$$

has property Q (2.3).

One then has

3.2. Theorem B_n . If a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ between small categories has property B_n ($n \geq 1$), then, for every object $Y \in \mathbf{Y}$, the category $(f\mathbf{X} \downarrow_n Y)$ (2.5(iv)) is a homotopy fibre of f over Y .

Closely related to property B_n is

3.3. Property C_n . Let \mathcal{O} denote the category consisting of a single object and its identity map and let n be an integer ≥ 1 . Then a small category \mathbf{Y} is said to have **property C_n** if

- (i) every functor $e: \mathcal{O} \rightarrow \mathbf{Y}$ has property B_n , i.e.
- (ii) every functor $e: \mathcal{O} \rightarrow \mathbf{Y}$ gives rise to a functor $(e\mathcal{O} \downarrow_n -): \mathbf{Y} \rightarrow \mathbf{Cat}$ which has property Q (2.3).

The usefulness of this notion is due to the fact that, in view of 2.2(ii), 2.6(ii) and 3.3(ii), one has

3.4. Theorem C_n . *If $f: \mathbf{X} \rightarrow \mathbf{Y}$ is a functor between small categories and \mathbf{Y} has property C_n ($n \geq 1$), then f has property B_n .*

We end with formulating our ultimate aim, namely

3.5. Theorem B_n for homotopy pullbacks. *Let n be an integer ≥ 1 and let $f: \mathbf{X} \rightarrow \mathbf{Y} \leftarrow \mathbf{Z}: g$ be a zigzag between small categories. If f has property B_n (3.1) (and in particular if \mathbf{Y} has property C_n (3.3)), then*

- (i) *the category $(f\mathbf{X} \downarrow_n g\mathbf{Z})$ (2.7) is a homotopy pullback of this zigzag.*

Moreover if in addition the monomorphism (2.7(iii))

$$k: (\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}) \longrightarrow (f\mathbf{X} \downarrow_n g\mathbf{Z})$$

is a weak equivalence, then

- (ii) *the pullback $(\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z})$ of this zigzag is also a homotopy pullback.*

4. THE PROOFS

It remains to give a proof of theorems 3.2 and 3.5, starting with

4.1. A proof of Theorem B_n (3.2). Given an object $Y \in \mathbf{Y}$, it follows from 2.4 and 2.6(i) that

- (i) $(f\mathbf{X} \downarrow_n Y)$ is the fibre as well as a homotopy fibre over Y of the projection functor

$$\pi: \mathbf{Gr}(f\mathbf{X} \downarrow_n -) = (f\mathbf{X} \downarrow_n \mathbf{Y}) \longrightarrow \mathbf{Y}.$$

That it is also a homotopy fibre over Y of the functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ therefore is a consequence of

- (ii) the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{h} & (f\mathbf{X} \downarrow_n \mathbf{Y}) \\ & \searrow f & \swarrow \pi \\ & \mathbf{Y} & \end{array}$$

in which h is as in 2.5(iii)

and the readily verifiable fact that

- (iii) h is a weak equivalence.

Finally we are ready to give

4.2. A proof of Theorem B_n for homotopy pullbacks (3.5). As the functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ has property B_n , i.e.

- the functor $(f\mathbf{X} \downarrow_n -): \mathbf{Y} \rightarrow \mathbf{Cat}$ has property Q (2.3)

it readily follows that

- the functor $(f\mathbf{X} \downarrow_n g-): \mathbf{Z} \rightarrow \mathbf{Cat}$ (2.8) also has property Q .

Consequently (2.4 and 2.8)

- for every object $Z \in \mathbf{Z}$, $(f\mathbf{X} \downarrow_n gZ)$ is the fibre as well as the homotopy fibre over Z of the projection functor

$$Gr(f\mathbf{X} \downarrow_n g-) = (f\mathbf{X} \downarrow_n gZ) \longrightarrow Z$$

Now consider the commutative square

$$\begin{array}{ccc} (f\mathbf{X} \downarrow_n \mathbf{Y}) & \xleftarrow{g'} & (f\mathbf{X} \downarrow_n gZ) \\ \pi \downarrow & & \downarrow \pi \\ \mathbf{Y} & \xleftarrow{g} & \mathbf{Z} \end{array}$$

in which g' is induced by g .

Then clearly

- this square is a pullback square and hence, for every object $Z \in \mathbf{Z}$, g' maps the fibre over Z isomorphically onto the fibre over $gZ \in \mathbf{Y}$.

Therefore, in view of (i) and 4.1(i)

- this pullback square is a homotopy pullback square.

With other words $(f\mathbf{X} \downarrow_n gZ)$ is a homotopy pullback of the zigzag

$$\pi: (f\mathbf{X} \downarrow_n \mathbf{Y}) \longrightarrow \mathbf{Y} \longleftarrow \mathbf{Z} : g$$

That it is also a homotopy pullback of the zigzag

$$f: \mathbf{X} \longrightarrow \mathbf{Y} \longleftarrow \mathbf{Z} : g$$

now follows from 4.1(ii) and 4.1(iii).

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